THE JORDANIAN DEFORMATION OF SU(2) AND CLEBSCH-GORDAN COEFFICIENTS†

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Representation theory for the Jordanian quantum algebra $\mathcal{U}_h(\mathrm{sl}(2))$ is developed using a nonlinear relation between its generators and those of sl(2). Closed form expressions are given for the action of the generators of $\mathcal{U}_h(\mathrm{sl}(2))$ on the basis vectors of finite dimensional irreducible representations. In the tensor product of two such representations, a new basis is constructed on which the generators of $\mathcal{U}_h(\mathrm{sl}(2))$ have a simple action. Using this basis, a general formula is obtained for the Clebsch-Gordan coefficients of $\mathcal{U}_h(\mathrm{sl}(2))$. Some remarkable properties of these Clebsch-Gordan coefficients are derived.

1 Introduction

The group GL(2) admits, upto isomorphism, only two quantum group deformations with central determinant : $\mathrm{GL}_q(2)$ and $\mathrm{GL}_h(2)$, see [1]. The quantum group $\mathrm{GL}_q(2)$ has been well studied, being the prototype example for many works on quantum groups. Investigations of the Jordanian quantum group $\mathrm{GL}_h(2)$, or $\mathrm{SL}_h(2)$, and its dual quantum algebra $\mathcal{U}_h(\mathrm{sl}(2))$ started more recently. Its defining relations were given in [2,3], and a construction of the dual Hopf algebra in [4]. Recently, also for the 2-parameter Jordanian quantum group $\mathrm{GL}_{g,h}(2)$ its dual was constructed [5]. For a development of its differential calculus or differential geometry we refer to [6] and [7]. A construction of the universal R-matrix was given in [8,9,10].

In this paper we are primarily interested in the irreducible finite dimensional representations of $\mathcal{U}_h(\mathrm{sl}(2))$. Also here, there has been progress in recent years. In [11], a direct construction of these representations was given by factorising the Verma module. An important development was given by Abdesselam *et al* [12] : they gave a nonlinear relation between the generators of $\mathcal{U}_h(\mathrm{sl}(2))$ and the classical generators of $sl(2)$. As a consequence they obtained expressions for the action of the generators of $\mathcal{U}_h(\mathrm{sl}(2))$ on basis vectors of the finite dimensional irreducible representations. These expressions were not always in closed form, and this was solved in [13]. In [14], finite and infinite dimensional representations of $\mathcal{U}_h(\mathrm{sl}(2))$ are constructed, and for the first time the tensor product of two representations is

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considered, yielding some examples of Clebsch-Gordan coefficients. The problem of determining Clebsch-Gordan coefficients was then completely solved in [13].

In the present paper we shall discuss a number of interesting properties of the Clebsch-Gordan coefficients of $\mathcal{U}_h(\mathrm{sl}(2))$, after recalling some of the main results of [13].

2 $\mathrm{SL}_h(2)$ and $\mathcal{U}_h(\mathrm{sl}(2))$

Consider the bialgebra $\mathcal{A}_h(2)$ with parameter h and four generators a, b, c, d subject to the relations :

$$\begin{array}{ll} ba = ab - ha^2 + h\mathcal{D} & ca = ac + hc^2 \\ da = ad + hdc - hac & bd = db - hd^2 + h\mathcal{D} \\ cd = dc + hc^2 & cb = bc + hdc + h^2c^2 \end{array} \tag{1}$$

where $\mathcal{D} = ad - bc - hac$. It is easy to verify the $\mathcal{D}$ is central. With $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, there is a comultiplication given by $\Delta(t) = t\dot{\otimes}t$, and a co-unit $\epsilon(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, turning $\mathcal{A}_h(2)$ into a coalgebra. The element $\mathcal{D}$ is group-like, so one can extend $\mathcal{A}_h(2)$ by $\mathcal{D}^{-1}$, and then an antipode S can be defined leading to the Hopf algebra $\mathrm{GL}_h(2)$. Putting $\mathcal{D} = 1$ gives rise to the matrix quantum group $\mathrm{SL}_h(2)$, see [11].

The dual Hopf algebra of $\mathrm{SL}_h(2)$ is denoted by $\mathcal{U}_h(\mathrm{sl}(2))$. It is an associative algebra generated by H, Y, T and T^{-1} satisfying quadratic relations [4]. For us it is more convenient to work with $X = (\log T)/h$, i.e. $T = e^{hX}$ and $T^{-1} = e^{-hX}$. Then the relations read :

$$\begin{array}{ll} [H, X] = 2\dfrac{\sinh hX}{h}, & [X, Y] = H, \\ [H, Y] = -Y(\cosh hX) - (\cosh hX)Y. & \end{array} \tag{2}$$

The comultiplication is given by :

$$\begin{aligned} \Delta(H) &= H \otimes e^{hX} + e^{-hX} \otimes H, \\ \Delta(X) &= X \otimes 1 + 1 \otimes X, \\ \Delta(Y) &= Y \otimes e^{hX} + e^{-hX} \otimes Y. \end{aligned} \tag{3}$$

The other ingredients (co-unit, antipode) are also defined, but not needed here.

3 Relation between $\mathcal{U}_h(\mathrm{sl}(2))$ and $\mathrm{sl}(2)$, and representations

With the following definition [12]

$$\begin{aligned} Z_+ &= \frac{2}{h} \tanh \frac{hX}{2}, \\ Z_- &= (\cosh \frac{hX}{2}) Y (\cosh \frac{hX}{2}), \end{aligned} \tag{4}$$

it follows that the elements $\{H, Z_+, Z_-\}$ satisfy the commutation relations of a classical sl(2) basis :

$$[H, Z_\pm] = \pm 2 Z_\pm, \qquad [Z_+, Z_-] = H.$$

These relations can be inverted, e.g.

$$e^{hX} = (1 + \frac{h}{2} Z_+)(1 - \frac{h}{2} Z_+)^{-1}.$$

These relations can also be used to give explicit matrix elements for the finite dimensional representations of $\mathcal{U}_h(\mathrm{sl}(2))$.

Recall that finite dimensional irreducible representations of sl(2) are labeled by a number j, with $2j$ a non-negative integer. The representation space can be denoted by $V^{(j)}$ with basis e^j_m $(m = -j, -j+1, \ldots, j)$, and the action is

$$\begin{aligned} &H e^j_m = 2m \, e^j_m, \\ &Z_\pm e^j_m = \sqrt{(j \mp m)(j \pm m + 1)} \, e^j_{m\pm 1}. \end{aligned} \tag{5}$$

For us, a more convenient basis for computations is the following v-basis related to the above e-basis by :

$$v^j_m = \alpha_{j,m} e^j_m, \;\; \text{with } \alpha_{j,m} = \sqrt{(j+m)!/(j-m)!} \, .$$

The sl(2) matrix elements in this basis are :

$$\begin{aligned} &H v^j_m = 2m \, v^j_m, \\ &Z_+ v^j_m = v^j_{m+1}, \\ &Z_- v^j_m = (j+m)(j-m+1) \, v^j_{m-1}., \end{aligned} \tag{6}$$

where $v^j_{j+1} \equiv 0$.

Using the explicit mapping between $\{H, Z_+, Z_-\}$ and $\{H, X, Y\}$, plus a number of combinatorial identities [13], we obtained :

Proposition 1 *The action of the generators of* $\mathcal{U}_h(\mathrm{sl}(2))$ *on the representation space* $V^{(j)}$ *is given by*

$$\begin{aligned} &H v^j_m = 2m \, v^j_m, \\ &X v^j_m = \sum_{k=0}^{\lfloor (j-m-1)/2 \rfloor} \frac{(h/2)^{2k}}{2k+1} \, v^j_{m+1+2k}, \\ &Y v^j_m = (j+m)(j-m+1) v^j_{m-1} - (j-m)(j+m+1) \left(\frac{h}{2}\right)^2 v^j_{m+1} \\ &\qquad + \sum_{s=1}^{\lfloor (j-m+1)/2 \rfloor} \left(\frac{h}{2}\right)^{2s} v^j_{m-1+2s}, \end{aligned} \tag{7}$$

It should be noted that the matrix elements of X were already obtained in [12]. Those of Y were also determined in [12], however not in closed form but as a complicated sum. In [13] we showed how such sums can be reduced to a simple form, using recently developed algorithms [15]. Proposition 1 is easy to apply and gives immediately all matrix elements of the $\mathcal{U}_h(\mathrm{sl}(2))$ generators. For example, the representatives for X and Y, respectively, in the v-basis for $j = 2$ are given by :

$$\begin{pmatrix} 0 & 1 & 0 & h^2/12 & 0 \\ 0 & 0 & 1 & 0 & h^2/12 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -3h^2/4 & 0 & h^4/16 & 0 \\ 4 & 0 & -5h^2/4 & 0 & h^4/16 \\ 0 & 6 & 0 & -5h^2/4 & 0 \\ 0 & 0 & 6 & 0 & -3h^2/4 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix};$$

with H given by the usual matrix $\mathrm{diag}(4, 2, 0, -2, -4)$. Note that the sl(2) representatives in the v-basis are recovered simply by putting $h = 0$.

4 Tensor product of $\mathcal{U}_h(\mathrm{sl}(2))$ representations

Consider $V^{(j_1)} \otimes V^{(j_2)}$ with basis $v^{j_1}_{m_1} \otimes v^{j_2}_{m_2}$. Our purpose is to show that this decomposes into the direct sum of representations $V^{(j)}$, $j = |j_1 - j_2|, \ldots, j_1 + j_2$. Note that the vectors $v^{j_1}_{m_1} \otimes v^{j_2}_{m_2}$ are in general no eigenvectors of $\Delta(H)$, since the comultiplication is given by :

$$\begin{aligned} \Delta(H) &= H \otimes e^{hX} + e^{-hX} \otimes H \\ &= H \otimes 1 + 1 \otimes H + 2H \otimes \sum_{n=1}^{\infty} \left(\frac{hZ_+}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{-hZ_+}{2}\right)^n \otimes 2H. \end{aligned} \tag{8}$$

The eigenvectors of $\Delta(H)$ are linear combinations of the vectors $v^{j_1}_{m_1} \otimes v^{j_2}_{m_2}$, and the coefficients play a crucial role in this work. To define these coefficients, recall the definition of the Pochhammer symbol :

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) & \text{if} \quad n = 1, 2, \ldots; \\ 1 & \text{if} \quad n = 0. \end{cases} \tag{9}$$

Next we define

$$b^{m_1,m_2}_{k,l} = \begin{cases} \dfrac{(-2m_1 - k)_l(-2m_2 - l)_k}{k!l!} & \text{if } k \geq 0 \text{ and } l \geq 0; \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

and finally the essential coefficients :

$$a^{m_1,m_2}_{k,l} = (-1)^k (h/2)^{k+l} (b^{m_1,m_2}_{k,l} - b^{m_1,m_2}_{k-1,l-1}). \tag{11}$$

Then we have the following important result :

Proposition 2 *In $V^{(j_1)} \otimes V^{(j_2)}$, the vectors*

$$w^{j_1,j_2}_{m_1,m_2} = \sum_{k=0}^{j_1-m_1} \sum_{l=0}^{j_2-m_2} a^{m_1,m_2}_{k,l} \; v^{j_1}_{m_1+k} \otimes v^{j_2}_{m_2+l} \tag{12}$$

form a basis consisting of eigenvectors of $\Delta(H)$. The explicit action of $\Delta(H)$, $\Delta(X)$ and $\Delta(Y)$ is given by

$$\begin{aligned}
&\Delta(H)\, w^{j_1,j_2}_{m_1,m_2} = 2(m_1+m_2)\, w^{j_1,j_2}_{m_1,m_2},\\
&\Delta(Z_+)\, w^{j_1,j_2}_{m_1,m_2} = w^{j_1,j_2}_{m_1+1,m_2} + w^{j_1,j_2}_{m_1,m_2+1},\\
&\Delta(Z_-)\, w^{j_1,j_2}_{m_1,m_2} = (j_1+m_1)(j_1-m_1+1)\, w^{j_1,j_2}_{m_1-1,m_2} +\\
&\qquad\qquad (j_2+m_2)(j_2-m_2+1)\, w^{j_1,j_2}_{m_1,m_2-1}.
\end{aligned} \tag{13}$$

Remark 3 This proposition tells us that the action of $\Delta(H)$, $\Delta(X)$ and $\Delta(Y)$ on the w-vectors is the same as the action of the su(2) generators (under the trivial Lie algebra comultiplication) on the uncoupled vectors $v^{j_1}_{m_1} \otimes v^{j_2}_{m_2}$. This observation implies the results on the tensor product decomposition and Clebsch-Gordan coefficients for $\mathcal{U}_h(\mathrm{sl}(2))$. In particular, the Clebsch-Gordan coefficients for $\mathcal{U}_h(\mathrm{sl}(2))$ are essentially given by linear combinations of su(2) Clebsch-Gordan coefficients, with $a^{m_1,m_2}_{k,l}$ the coefficients of this linear combination.

Let us first consider an example, say $V^{(1)} \otimes V^{(1/2)}$. Using the formulas (10)-(12), the w-vectors are explicitly given by

$$\begin{pmatrix} w^{1,1/2}_{-1,-1/2} \\ w^{1,1/2}_{-1,1/2} \\ w^{1,1/2}_{0,-1/2} \\ w^{1,1/2}_{0,1/2} \\ w^{1,1/2}_{1,-1/2} \\ w^{1,1/2}_{1,1/2} \end{pmatrix} = \begin{pmatrix} 1 & h & -h/2 & h^2/4 & h^2/4 & -h^3/8 \\ 0 & 1 & 0 & h/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -h/2 & h^2/4 \\ 0 & 0 & 0 & 1 & 0 & h/2 \\ 0 & 0 & 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^1_{-1} \otimes v^{1/2}_{-1/2} \\ v^1_{-1} \otimes v^{1/2}_{1/2} \\ v^1_0 \otimes v^{1/2}_{-1/2} \\ v^1_0 \otimes v^{1/2}_{1/2} \\ v^1_1 \otimes v^{1/2}_{-1/2} \\ v^1_1 \otimes v^{1/2}_{1/2} \end{pmatrix}.$$

It is easy to verify that the inverse of the above upper-triangular matrix is given by reflecting the matrix along its second diagonal, i.e. by its skew-transpose :

$$\begin{pmatrix} 1 & -h & h/2 & h^2/4 & 0 & -h^3/8 \\ 0 & 1 & 0 & -h/2 & 0 & h^2/4 \\ 0 & 0 & 1 & 0 & h/2 & h^2/4 \\ 0 & 0 & 0 & 1 & 0 & -h/2 \\ 0 & 0 & 0 & 0 & 1 & h \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This turns out to be a general property of these matrices of $a^{m_1,m_2}_{k,l}$ coefficients. In other words, we have

Proposition 4 *The coefficients* $a_{k,l}^{m_1,m_2}$ *satisfy*

$$\sum_{n_1,n_2} a_{n_1-m_1,n_2-m_2}^{m_1,m_2} a_{M_1-n_1,M_2-n_2}^{-M_1,-M_2} = \delta_{m_1,M_1}\delta_{m_2,M_2}. \tag{14}$$

Note that the above formula is nontrivial only for $M_1 \geq m_1$ and $M_2 \geq m_2$, otherwise the indices of the a-coefficients are negative and thus automatically zero. The above property follows from the following remarkable identity holding for arbitrary parameters x and y :

$$\begin{aligned} &\sum_{k=0}^{K}\sum_{l=0}^{L} \frac{(-x-k)_l(-y-l)_k}{k!l!} \frac{(x+K+k)_{L-l}(y+L+l)_{K-k}}{(K-k)!(L-l)!} \frac{(xy+lx+ky)}{(x+k)(y+l)} \\ &\times \frac{(xy+Lx+Ky+lx+ky+2Kl+2kL)}{(x+K+k)(y+L+l)} = \delta_{K,0}\delta_{L,0}, \end{aligned} \tag{15}$$

by putting $x = 2m_1$, $y = 2m_2$, $K = M_1 - m_1$ and $L = M_2 - m_2$. The proof of (15) falls beyond the scope of the present paper.

5 Clebsch-Gordan coefficients and properties

From Remark 3 it is easy to deduce that the decomposition of the tensor product is given by

$$V^{(j_1)} \otimes V^{(j_2)} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V^{(j)},$$

and we have

Proposition 5 *The Clebsch-Gordan coefficients for* $\mathcal{U}_h(\mathrm{sl}(2))$*, in*

$$e_m^{(j_1 j_2)j} = \sum_{n_1,n_2} \mathcal{C}_{n_1,n_2,m}^{j_1,j_2,j}(h)\; e_{n_1}^{j_1} \otimes e_{n_2}^{j_2},$$

are given by

$$\mathcal{C}_{n_1,n_2,m}^{j_1,j_2,j}(h) = \sum_{m_1+m_2=m} C_{m_1,m_2,m}^{j_1,j_2,j} A_{n_1-m_1,n_2-m_2}^{m_1,m_2},$$

with $C_{m_1,m_2,m}^{j_1,j_2,j}$ *the usual* su(2) *Clebsch-Gordan coefficients, and* $A_{n_1-m_1,n_2-m_2}^{m_1,m_2}$ *determined by*

$$A_{k,l}^{m_1,m_2} = a_{k,l}^{m_1,m_2} \frac{\alpha_{j_1,m_1+k}\alpha_{j_2,m_2+l}}{\alpha_{j_1,m_1}\alpha_{j_2,m_2}}.$$

So apart from the α-factors (which appear here because we have formulated the proposition in the e-basis rather than in the v-basis), the Clebsch-Gordan matrix is essentially the product of the corresponding su(2) Clebsch-Gordan matrix with the upper triangular matrix of a-coefficients considered in the previous section.

From the explicit form of the a-coefficients, and Proposition 5, it follows that

Proposition 6 *The Clebsch-Gordan coefficients of* $\mathcal{U}_h(\mathrm{sl}(2))$ *satisfy*

- *if* $m = n_1 + n_2$ *then* $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h) = C^{j_1,j_2,j}_{n_1,n_2,m}$ *;*
- *if* $m > n_1 + n_2$ *then* $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h) = 0$*;*
- *if* $m < n_1 + n_2$ *then* $\mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h)$ *is a monomial in* $h^{n_1+n_2-m}$*.*

The most interesting property follows from Proposition 4 :

Proposition 7 *The Clebsch-Gordan coefficients of* $\mathcal{U}_h(\mathrm{sl}(2))$ *satisfy the skew-orthogonality relations*

$$\sum_{n_1,n_2} (-1)^{j_1+j_2-j} \mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h) \mathcal{C}^{j_1,j_2,j'}_{-n_1,-n_2,-m'}(h) = \delta_{j,j'} \delta_{m,m'},$$
$$\sum_{j,m} (-1)^{j_1+j_2-j} \mathcal{C}^{j_1,j_2,j}_{n_1,n_2,m}(h) \mathcal{C}^{j_1,j_2,j}_{-n'_1,-n'_2,-m}(h) = \delta_{n_1,n'_1} \delta_{n_2,n'_2}.$$

This property gives in fact the inverse matrix of a general Clebsch-Gordan matrix of $\mathcal{U}_h(\mathrm{sl}(2))$. The proof is as follows : recall that the Clebsch-Gordan matrix of $\mathcal{U}_h(\mathrm{sl}(2))$ is essentially the product of an upper-triangular matrix of a-coefficients with an su(2) Clebsch-Gordan matrix. But the upper-triangular matrix has an easy inverse, namely its skew-transpose; and also the su(2) Clebsch-Gordan matrix has an easy inverse, namely its transpose (since it is orthogonal). This, and some symmetry properties of su(2) Clebsch-Gordan coefficients, leads to Proposition 7.

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